

# Scalar-Tensor Gravity Cosmology: Noether symmetries and analytical solutions

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In this paper, we present a complete Noether Symmetry analysis in the framework of scalar-tensor cosmology. Specifically, we consider a non-minimally coupled scalar field action embedded in the Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime and provide a full set of Noether symmetries for related minisuperspaces. The presence of symmetries implies that the dynamical system becomes integrable and then we can compute cosmological analytical solutions for specific functional forms of coupling and potential functions selected by the Noether Approach.

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## 1. INTRODUCTION

The discovery of the accelerated expansion of the universe [1–8] has opened a new path in approaching the cosmological problem. Despite the mounting observational evidences on the existence of the cosmic acceleration, its nature and fundamental origin is still an open question challenging the very foundations of theoretical physics. Usually, the mechanism that is responsible for cosmic acceleration is attributed to new physics which is based either on a modified theory of gravity or on the existence of some sort of dark energy which is associated with new fields in nature (see [9–31] and references therein).

From the mathematical viewpoint, in order to study the cosmological features of a particular “dark energy” model, it is essential to specify the covariant Einstein-Hilbert action of the model and find out the corresponding energy-momentum tensor. This methodology provides an elegant way to deal with dark energy in cosmology. Within this framework, the standard view of the classical scalar field dark energy can be generalized considering scalar-tensor theories of gravity in which the scalar field  $\psi$  is non-minimally coupled to the Ricci scalar  $R$ . Generally, any theory of gravity that is not simply linear in the Ricci scalar can be reduced to a scalar-tensor one, implying that among the modified gravity models the scalar-tensor theory of gravity is one of the most general case that contains also other alternatives (for a review see [31]). As an example, the  $f(R)$ -gravity can be seen as a particular case of scalar-tensor gravity obeying the following criteria: (a) the scalar field is non-minimally coupled to the Ricci scalar and (b) a self-interacting potential is present while there is no kinetic term. In this specific case, the scalar field is  $\psi = f'(R)$  which is the first derivative of  $f(R)$  function with respect to  $R$ . In general, large classes of alternative theories of gravity, non-linear in the curvature invariants or non-minimally

coupled in the Jordan frame, can be reduced to general relativity plus scalar field(s) in the Einstein frame [12].

In a recent paper by the same authors [32], conformally related metrics and Lagrangians, in the framework of scalar-tensor cosmology, have been studied. In particular, it has been proven that the field equations of two conformally related Lagrangians are also conformally related if the corresponding Hamiltonian vanishes. This is an important feature strictly related to the energy conditions of the theory. Also, it has been shown that to every non-minimally coupled scalar field, we can associate a unique minimally coupled scalar field in a conformally related space with an appropriate potential. The existence of such a connection can be used in order to study the dynamical properties of the various cosmological models, since the field equations of a non-minimally coupled scalar field can be reduced, at conformal level, to the field equations of the minimally coupled scalar field.

With the current work, we complete our previous program on scalar-tensor theories by calculating the corresponding Noether point symmetries as well as the related analytical solutions. It is interesting to mention that Noether point symmetries have gained a lot of attention in cosmology (see [33–47]), since they can be used as a selection criterion in order to discriminate the dark energy models, including those of modified gravity [44] as well as to provide analytical solutions. Such a program started in [34] where inflationary models have been considered. The paradigm can be shortly summarized as follows. The existence of a Noether symmetry selects the forms of non-minimal coupling and potential in general scalar-tensor theories of gravity. As a consequence, the related dynamical system which results *reduced* because every symmetry is related to a first integral of motion. In most cases, the presence of such integrals of motion allows to find out general solutions for dynamics. It is important to stress that by choosing particular classes

of metrics, one reduces the field theory to a point-like one. From a cosmological viewpoint, this means that we are considering dynamical systems defined on *minisuperspaces*. These finite-dimensional dynamical systems are extremely interesting in Quantum Cosmology (see [43] for a discussion). This consideration is important since allows one to deal with Noether Symmetry Approach both in early and late cosmology.

Some remarks are important at this point to relate the Noether Symmetry Approach to the presence of conserved physical quantities. Generally speaking, in modified gravitational theories, where the Birkhoff theorem is not guaranteed, the Noether approach can provide a useful tool towards describing the global dynamics [48], through the first integrals of motion. Moreover, besides the technical possibility of reducing the dynamical system, the first integrals of motion give always rise to conserved currents that are not only present in physical space-time but also in configuration spaces (see the discussion in [32] and [34]). While in space-time such currents are linear momentum, angular momentum etc. in configuration space the conserved quantities emerge as relations among dynamical variables, in particular, among their functions as couplings and self-interaction potentials. For example, as discussed in Capozziello & Ritis [49], the presence of Noether symmetries in scalar-tensor gravity gives rise to an effective cosmological constant and gravitational asymptotic freedom behaviours induced by potentials and couplings. This means that, while in the standard spacetime the Noether charges are directly related to conserved observable quantities, in the configuration space (minisuperspace), they are present as "selection rules" for potentials and coupling functions which are capable of assigning realistic dynamics.

In the present work, we complete the program started in [34] and [32], discussing the general structure of scalar-tensor cosmological models compatible with the existence of Noether symmetries. Moreover, the current work can be seen as a natural continuation of our previous works [44]. The layout of the paper is the following. In Sec. 2, we present the main ingredients of the dynamical problem under study. In Secs. 3 and 4 we provide the Noether point symmetries as well as the corresponding analytical solutions for the two classes of models considered. We draw our conclusions in Sec. 5.

## 2. THE MINISUPERSPACE AND THE DYNAMICAL SYSTEM

In the context of scalar-tensor cosmology, let us consider a scalar field  $\psi$  (non-minimally) interacting with the gravitational field. In this framework, the field equations can be derived from the following general action

$$S = \int dt dx^3 \sqrt{-g} \left[ F(\psi, R) + \frac{\varepsilon}{2} g_{ij} \psi^{;i} \psi^{;j} - V(\psi) \right] + S_m \quad (1)$$

where  $\varepsilon = \pm 1$ ,  $\psi$  denotes the scalar field,  $V(\psi)$  is the self-interaction potential,  $F(\psi, R)$  is the coupling function,  $R$  is the Ricci scalar and  $S_m$  is the matter action. The parameter  $\varepsilon$  indicates if we are dealing with a regular scalar field or a ghost field. Assuming a spatially flat FRW space-time

$$ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j,$$

the infinite degrees of freedom of the field theory reduce to a finite number. In this specific case, the minisuperspace is a 2-dimensional configuration space defined by the variables  $\mathcal{Q} = \{\psi, a\}$ . The tangent space on which dynamics is defined is  $\mathcal{TQ} = \{\psi, \dot{\psi}, a, \dot{a}\}$  where the dot indicates the derivative with respect to the cosmic time which is the natural affine parameter for the problem. Of course, if we consider  $F(\psi, R) = R$  then the action (1) boils down to the nominal, minimally coupled, scalar field dark energy. On the other hand, the  $f(R)$  modified gravity is fully recovered for  $F(\psi, R) = f(R)$  and in the absence of the kinetic term in the action. In this study, we consider the case where the coupling function is proportional to  $R$ ,  $F(\psi, R) = F(\psi)R$ .

Due to the fact that almost every dynamical system is described by a corresponding Lagrangian, below we apply such ideas to the scalar field cosmology. Indeed the corresponding Lagrangian and the Hamiltonian (total energy density) of the field equations are

$$L = 6F(\psi) a \dot{a}^2 + 6F_\psi(\psi) a^2 \dot{\psi} + \frac{\varepsilon}{2} a^3 \dot{\psi}^2 - a^3 V(\psi) \quad (2)$$

$$E = 6F(\psi) a \dot{a}^2 + 6F_\psi(\psi) a^2 \dot{\psi} + \frac{\varepsilon}{2} a^3 \dot{\psi}^2 + a^3 V(\psi) \quad (3)$$

Note that the Lagrangian (2) is autonomous, hence the Hamiltonian  $E$  is a constant of motion (see also the discussion in [32]). This constant corresponds to the trivial Noether point symmetry  $\partial_t$  (first integral of motion).

Using the 00 component of the conservation equation  $T_{;\mu}^{\nu\mu} = 0$  we find that the Hamiltonian  $E$  is related to the matter density  $\rho_m$  as  $\rho_m = \frac{|E|}{a^3}$ .

Following the technique described in [44, 50, 51] it is essential to split the Lagrangian (2) in the kinematic part, which defines the kinematic metric (hereafter KM), and the remaining part which we consider to be the potential. Indeed the kinematic metric is written as

$$ds_{KM}^2 = 12F(\psi) a \dot{a}^2 + 12F_\psi(\psi) a^2 \dot{\psi} + \varepsilon a^3 \dot{\psi}^2 \quad (4)$$

The above metric is not the FRW metric of the background space-time but a metric defined on the tangent space  $\mathcal{TQ}$ . It is related to the minisuperspace configuration metric in the two dimensional space  $\{a, \psi\}$ . The corresponding Ricci scalar of the metric  $ds_{KM}^2$  is computed to be:

$$R_{KM} = \frac{\varepsilon}{4a^3} \frac{(2F_{\psi\psi} F - F_\psi^2)}{(\varepsilon F - 3F_\psi^2)^2} \quad (5)$$

Obviously, knowing  $R_{KM}$ , one can estimate  $F(\psi)$ . If we assume that the curvature  $R_{KM}$  is constant then Eq.(5) implies that  $R_{KM} \equiv 0$  (due to the presence of  $a$  in the denominator) and the minisuperspace is flat. We realize that we need to consider the following two cases: (A) the minisuperspace  $\{a, \psi\}$  is maximal symmetric (flat  $R_{KM} \equiv 0$ ) and (B) the case where the minisuperspace is not necessarily flat but it is conformally flat, because all two dimensional spaces are conformally flat. In the following, we consider these two situations in detail.

### 3. THE CASE OF MAXIMALLY SYMMETRIC $\{a, \psi\}$ MINISUPERSPACE

In this case using the condition  $R_{KM} = 0$ , Eq.(5) reduces to

$$2F_{\psi\psi}F - F_{\psi}^2 = 0 \quad (6)$$

and then a solution is

$$F(\psi) = -\frac{F_0\varepsilon}{12}(\psi + \psi_0)^2 \quad (7)$$

where  $F_0\varepsilon > 0$ .

In order to determine the homothetic algebra of the kinematic metric (4), we write it in a more familiar form. Actually, in Tsamparlis et al. [32] we introduced the conformal variables  $A$ ,  $\Psi$  and  $\mathcal{N}$  by the relations

$$A = \sqrt{-2Fa} \quad (8)$$

$$d\Psi = \sqrt{\left(\frac{3\varepsilon F_{\psi}^2 - F}{2F^2}\right)} d\psi \quad (9)$$

$$\mathcal{N} = \frac{1}{\sqrt{-2F}} \quad (10)$$

with  $F(\Psi) < 0$ . In the new variables, the kinematic metric (4) and the Lagrangian (2) become

$$ds_{KM}^2 = N^2(\Psi) \left[ -3A\dot{A}^2 + \frac{\varepsilon}{2}A^3\dot{\Psi}^2 \right] \quad (11)$$

$$L = N^2(\Psi) \left[ -3A\dot{A}^2 + \frac{\varepsilon}{2}A^3\dot{\Psi}^2 \right] - A^3\bar{V}(\Psi) \quad (12)$$

where  $N^2 = \mathcal{N}$ ,  $\bar{V}(\Psi) = N^6(\Psi)V(\Psi)$ . Also, the coupling function (7) takes the form

$$F(\Psi) = -\frac{\varepsilon F_0}{12}e^{\sqrt{6\varepsilon}|k|\Psi} \quad (13)$$

where

$$|k| = \frac{1}{3}\sqrt{\frac{|F_0|}{|1 + \varepsilon F_0|}}. \quad (14)$$

Notice, that the inequality  $F_0\varepsilon > 0$  is satisfied either for  $\varepsilon = +1$  with  $F_0 > 0$  or for  $\varepsilon = -1$  with  $-1 < F_0 < 0$ . We would like to mention here that in the case of  $\varepsilon = -1$  with  $F_0 < -1$  one has to replace  $\Psi$  with  $i\Psi$ .

We further simplify the above calculations by introducing a new coordinate system  $(r, \theta)$  defined as

$$r = \sqrt{\frac{8}{3}}A^{\frac{3}{2}}, \quad \theta = \sqrt{\frac{3\varepsilon}{8}}\Psi. \quad (15)$$

Inserting the above variables into Eq.(11), we immediately obtain

$$ds_{KM}^2 = N^2(\theta) (-dr^2 + r^2 d\theta^2) \quad (16)$$

which is directly related to the flat 2D Lorentzian space with metric

$$ds^2 = -dr^2 + r^2 d\theta^2$$

with the conformal factor  $N(\theta)$

$$N^2(\theta) = N_0^2 e^{\mp 2|k|\theta} \quad N_0^2 = \left(\frac{6}{\varepsilon F_0}\right)^{1/2}. \quad (17)$$

Finally, the Lagrangian takes a rather simple form

$$L = N^2(\theta) \left( -\frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 \right) - r^2 V(\theta). \quad (18)$$

Armed with the above expressions, we can deduce the homothetic algebra of the metric from well known previous results (see [44, 50, 51]).

#### 3.1. Searching for Noether point symmetries

Let us determine now all the potentials  $V(\psi)$  for which the above dynamical system admits Noether point symmetries beyond the trivial one  $\partial_t$  related to the energy. Subsequently, we shall use the resulting Noether integrals in order to find out analytical solutions.

For  $|k| \neq 1$  the homothetic algebra consists of the gradient Killing vectors (KVs)

$$K^1 = \frac{e^{(1-k)\theta} r^k}{N_0^2} \left( -\partial_r + \frac{1}{r}\partial_\theta \right), \quad S_1(r, \theta) = \frac{r^{1+k} e^{(1+k)\theta}}{(k+1)}$$

$$K^2 = \frac{e^{-(1+k)\theta} r^{-k}}{N_0^2} \left( \partial_r + \frac{1}{r}\partial_\theta \right), \quad S_2(r, \theta) = \frac{r^{1-k} e^{-(1-k)\theta}}{k-1}$$

(for  $|k| = 1$  see Appendix A) the non-gradient KV

$$K^3 = r\partial_r - \frac{1}{k}\partial_\theta$$

and the gradient homothetic vectors (HV)

$$H^i = \frac{1}{N_0^2(k^2-1)} (-r\partial_r + k\partial_\theta), \quad H(r, \theta) = \frac{1}{2} \frac{r^2 e^{2k\theta}}{(k^2-1)}.$$

Specifically, we ask the question: *are there potentials that can provide non-trivial Noether point symmetries and consequently first integrals of motion?* Below we present all possible cases:

1. First of all, by using the gradient KV  $K^1$ , we find

a) for  $V(\theta) = V_0 e^{2\theta}$ , we have the Noether symmetries  $K^1$ ,  $tK^1$  with Noether integrals

$$I_1 = \frac{d}{dt} \left( \frac{r^{1+k} e^{(1+k)\theta}}{(k+1)} \right)$$

$$I_2 = t \frac{d}{dt} \left( \frac{r^{1+k} e^{(1+k)\theta}}{(k+1)} \right) - \left( \frac{r^{1+k} e^{(1+k)\theta}}{(k+1)} \right)$$

b) for  $V(\theta) = V_0 e^{2\theta} - \frac{mN_0^2}{2(k^2-1)} e^{2k\theta}$ , we obtain the Noether symmetries  $e^{\pm\sqrt{m}t} K^1$ , where  $m = \text{constant}$ , with Noether integrals

$$I'_\pm = e^{\pm\sqrt{m}t} \left[ \frac{d}{dt} \left( \frac{r^{1+k} e^{(1+k)\theta}}{(k+1)} \right) \mp \sqrt{m} \left( \frac{r^{1+k} e^{(1+k)\theta}}{(k+1)} \right) \right]$$

From the above Noether integrals, we construct the time independent first integral  $I_{K^1} = I_+ I_-$ .

2. The gradient KV  $K^2$  produces the Noether symmetries for the following potentials

a) for  $V(\theta) = V_0 e^{-2\theta}$ , we have the Noether symmetries  $K^1$ ,  $tK^1$  with Noether integrals

$$J_1 = \frac{d}{dt} \left( \frac{r^{1-k} e^{-(1-k)\theta}}{k-1} \right),$$

$$J_2 = t \frac{d}{dt} \left( \frac{r^{1-k} e^{-(1-k)\theta}}{k-1} \right) - \frac{r^{1-k} e^{-(1-k)\theta}}{k-1}$$

b) for  $V(\theta) = V_0 e^{-2\theta} - \frac{mN_0^2}{2(k^2-1)} e^{2k\theta}$ , we have the Noether symmetries  $e^{\pm\sqrt{m}t} K^2$   $m = \text{constant}$ , with Noether integrals

$$J'_\pm = e^{\pm\sqrt{m}t} \left[ \frac{d}{dt} \left( \frac{r^{1-k} e^{-(1-k)\theta}}{k-1} \right) \mp \sqrt{m} \frac{r^{1-k} e^{-(1-k)\theta}}{k-1} \right]$$

Combining the latter Noether integrals, we construct the time-independent first integral  $J_{K^2} = J'_+ J'_-$ .

3. The non gradient KV  $K^3$  produces a Noether symmetry for the potential  $V(\theta) = V_0 e^{2k\theta}$  with Noether integral

$$I_3 = \frac{r e^{2k\theta}}{k} (k\dot{r} + r\dot{\theta}).$$

4. The gradient HV produces the following Noether symmetries for the following potentials

a) for  $V(\theta) = V_0 e^{-2\frac{(k^2-2)}{k}\theta}$ ,  $k^2 - 2 \neq 0$  we have the Noether symmetries  $2t\partial_t + H^i$ ,  $t^2\partial_t + tH^i$  with Noether integrals

$$I_{H_1} = 2tE - \frac{d}{dt} \left( \frac{1}{2} \frac{r^2 e^{2k\theta}}{(k^2-1)} \right),$$

$$I_{H_2} = t^2 E - t \frac{d}{dt} \left( \frac{1}{2} \frac{r^2 e^{2k\theta}}{(k^2-1)} \right) + \frac{1}{2} \frac{r^2 e^{2k\theta}}{(k^2-1)}.$$

We note that in this case the system is the Ermakov-Pinney dynamical system [52] and admits the Noether symmetry algebra  $sl(2, R)$ .

b) For  $V(\theta) = V_0 e^{-2\frac{(k^2-2)}{k}\theta} - \frac{N_0^2 m}{k^2-1} e^{2k\theta}$ ,  $k^2 - 2 \neq 0$  we have the Noether symmetries  $\frac{2}{\sqrt{m}} e^{\pm\sqrt{m}t} \partial_t \pm e^{\pm\sqrt{m}t} H^i$ ,  $m = \text{constant}$  with Noether integrals

$$I_\pm = e^{\pm 2\sqrt{m}t} \left[ \frac{1}{\sqrt{m}} E \mp \frac{d}{dt} \left( \frac{1}{2} \frac{r^2 e^{2k\theta}}{(k^2-1)} \right) + 2\sqrt{m} \left( \frac{1}{2} \frac{r^2 e^{2k\theta}}{(k^2-1)} \right) \right]$$

This is also the Ermakov-Pinney dynamical system with a linear oscillator. Therefore it admits the Ermakov - Pinney invariant which we may construct with the use of the dynamical Noether symmetries or with the use of the corresponding Killing Tensor.

5. Lastly, the case  $V(\theta) = 0$  corresponds to the free particle (see [50]).

### 3.2. Analytical solutions

Using the above Noether symmetries and the corresponding integral of motions, we can fully solve the dynamical problem of the scalar tensor cosmology. In order to simplify the analytical solutions, we consider the new variables

$$x = S_1(r, \theta) = \frac{r^{1+k} e^{(1+k)\theta}}{k+1}, \quad y = S_2(r, \theta) = \frac{r^{1-k} e^{-(1-k)\theta}}{k-1} \quad (19)$$

and the inverse transformation is

$$\theta = \frac{1}{2|k^2-1|} \ln \left[ \frac{|k^2-1|^{1-k} x^{1-k}}{(k-1)^2 y^{1+k}} \right] \quad (20)$$

$$r = \sqrt{|k^2-1|xy} \left[ \frac{|k^2-1|^{1-k} x^{1-k}}{(k-1)^2 y^{1+k}} \right]^{\frac{k}{2(k^2-1)}}. \quad (21)$$

We find that in the new coordinates  $(x, y)$ , the Lagrangian (18) takes the form

$$L(x, y, \dot{x}, \dot{y}) = \epsilon_k \frac{N_0^2}{2} \dot{x}\dot{y} - U(x, y) \quad (22)$$

where  $U(x, y) = r^2 V(\theta)$  and  $\epsilon_k = +1$  for  $|k| > 1$  ( $\epsilon_k = -1$  for  $|k| < 1$ ). Note that  $V(\theta)$  are the potentials which have been presented in the previous section.

We would like to stress that the solution of the field equations for each potential is a formal and lengthy operation which adds nothing but unnecessary material to the matter. What is interesting of course is the final answer for each case and this is what we show in a compact presentation below. Specifically, the analytical solutions can be categorized into seven separate cases

- The first class is  $U_1(x, y) = V_0 r^2 e^{2k\theta} = V_0 |k^2 - 1| xy$

$$x(t) = x_1 \text{Sinn}(\omega t) + x_2 \text{Coss}(\omega t) \quad (23)$$

$$y(t) = y_1 \text{Sinn}(\omega t) + y_2 \text{Coss}(\omega t) \quad (24)$$

where

$$(\text{Sinn}\omega, \text{Coss}\omega) = \begin{cases} (\sin\omega, \cos\omega) & |k| > 1 \\ (\sinh\omega, \cosh\omega) & |k| < 1 \end{cases} \quad (25)$$

$\omega^2 = \frac{2V_0|k^2-1|}{N_0^2}$  and the Hamiltonian is

$$E = V_0 |k^2 - 1| (x_1 y_1 + \epsilon_k x_2 y_2)$$

- $U_2(x, y) = V_0 r^2 e^{2\theta} = V_0 (k+1)^{\frac{2}{1+k}} x^{\frac{2}{1+k}}$ , as long as  $k \neq -3$  we have

$$x(t) = x_1 t + x_2 \quad (26)$$

$$y(t) = -\epsilon_k \frac{2\bar{V}(k+1)(x_1 t + x_2)^{(1+\frac{2}{1+k})}}{x_1^2(3+k)N_0^2} + y_1 t + y_2 \quad (27)$$

where  $\bar{V} = V_0(k+1)^{\frac{2}{1+k}}$  and the Hamiltonian is

$$E = \epsilon_k \frac{y_1 x_1 N_0^2}{2}.$$

If  $k = -3$  then  $y(t)$  becomes

$$y(t) = -2 \frac{\bar{V}}{N_0^2 x_1^2} \ln(x_1 t + x_2) + y_1 t + y_2. \quad (28)$$

- $U_3(x, y) = V_0 r^2 e^{-2\theta} = V_0 |k-1|^{\frac{2}{1-k}} y^{\frac{2}{1-k}}$

When  $k \neq 3$

$$x(t) = \frac{2\bar{V}|k-1|(y_1 t + y_2)^{1+\frac{2}{1-k}}}{y_1^2(k-3)N_0^2} + x_1 t + x_2 \quad (29)$$

$$y(t) = y_1 t + y_2 \quad (30)$$

where  $\bar{V} = V_0 |k-1|^{\frac{2}{1-k}}$  and the Hamiltonian is

$$E = \epsilon_k \frac{y_1 x_1 N_0^2}{2}.$$

In this context if  $k = 3$  then  $x(t)$  takes the form

$$x(t) = -\frac{2\bar{V}}{N_0^2 y_1^2} \ln(y_1 t + y_2) + x_1 t + x_2. \quad (31)$$

- $U_4(x, y) = V_0 r^2 e^{2\theta} + m r^2 e^{2k\theta} = \bar{V}_0 x^{\frac{2}{1+k}} + \bar{m} xy$ , in this class we find

$$x(t) = x_1 \text{Sinn}(\omega t + \omega_0) \quad (32)$$

$$y(t) = \text{Coss}(\omega t + \omega_0) \left( y_1 + 2\epsilon_K \frac{\omega}{\bar{m}} \int \frac{E - x_1 \bar{V}_0 \text{Sinn}(\omega t + \omega_0)^{\frac{2}{1+k}}}{x_1 (\text{Coss}(\omega t + \omega_0) + 1)} dt \right) \quad (33)$$

where  $\bar{V}_0 = V_0(k+1)^{\frac{2}{1+k}}$ ,  $\bar{m} = m|k^2 - 1|$ ,  $\omega^2 = \frac{2\bar{m}}{N_0^2}$  and  $E = y_2$ .

- similarly for  $U_5(x, y) = r^2 e^{-2\theta} + m r^2 e^{2k\theta} =$

$\bar{V}_0 y^{\frac{2}{1-k}} + \bar{m} xy$  we obtain

$$x(t) = \text{Coss}(\omega t + \omega_0) \left( x_1 + 2\epsilon_k \frac{\omega}{\bar{m}} \int \frac{x_2 - y_1 \bar{V}_0 \text{Sinn}(\omega t + \omega_0)^{\frac{2}{1-k}}}{y_1 (\text{Coss}(\omega t + \omega_0) + 1)} dt \right) \quad (34)$$

$$y(t) = y_1 \text{Sinn}(\omega t + \omega_0) \quad (35)$$

where  $\bar{V}_0 = V_0 |k-1|^{\frac{2}{1-k}}$  and  $E = x_2$ .

- $U_6(x, y) = V_0 r^2 e^{-2\frac{(k^2-2)}{k}\theta} + m r^2 e^{2k\theta} =$

$\bar{V}_0 \frac{1}{y^2} \left( \frac{x}{y} \right)^{\frac{2}{k}-1} + \bar{m} xy$  with  $\bar{V}_0 = V_0 \frac{|k^2-1|^{\frac{2}{k}-1}}{|k-1|^{\frac{4}{k}}}$ .

The current dynamical system is the so called Ermakov-Pinney system. To solve this dynamical problem, it is convenient to go to the following con-

ordinates  $(x, y) = (ze^w, ze^{-w})$ . In this coordinate system we recover the Ermakov-Pinney equation:

$$\ddot{z} + 2\epsilon_k \bar{m} z + \epsilon_k N_0^2 \frac{J_{EL}}{z^3} = 0 \quad (36)$$

where  $J_{EL} = z^4 \dot{w}^2 - 2\epsilon_k \frac{\bar{V}_0}{N_0^2} e^{\frac{4}{k} w}$  is the Ermakov invariant. The solution of the above differential equation is

$$z(t) = [l_0 z_1(t) + l_1 z_2(t) + l_3]^{\frac{1}{2}} \quad (37)$$

$$e^{\frac{4}{k} w(t)} = -\epsilon_k \frac{N_0^2 J_{EL}}{2\bar{V}_0} \left[ 1 - \tanh^2 \left( \frac{2\sqrt{J_{EL}}}{k} \left( \int \frac{dt}{z^2(t)} + l_4 \right) \right) \right] \quad (38)$$

where  $z_{1,2}(t)$  are solutions of the differential equation  $\ddot{z} + 2\epsilon_k \bar{m} z = 0$  and  $l_{0-4}$  are constants.

- Lastly,  $U_7(x, y) = 0$  is the free particle system, a solution of which is

$$x(t) = x_1 t + x_2, \quad y(t) = y_1 t + y_2 \quad (39)$$

with  $E = \epsilon_k \frac{N_0^2}{2} x_1 y_1$ .

#### 4. THE CASE OF 2D CONFORMALLY-FLAT METRIC

In this case the kinetic metric (16) is non-flat (i.e.  $R_{KM} \neq 0$ ) but, of course, it is conformally flat being a two dimensional metric. Its conformal algebra is infinity dimensional; however it has a closed subalgebra consisting of the following vectors (this is the special conformal algebra of  $M^2$ ):

$$\begin{aligned} X^1 &= \cosh \theta \partial_r - \frac{1}{r} \sinh \theta \partial_\theta, \quad X^2 = \sinh \theta \partial_r - \frac{1}{r} \cosh \theta \partial_\theta \\ X^3 &= \partial_\theta, \quad X^4 = r \partial_r, \quad X^5 = \frac{1}{2} r^2 \cosh \theta \partial_r + \frac{1}{2} r \sinh \theta \partial_\theta \\ X^6 &= \frac{1}{2} r^2 \sinh \theta \partial_r + \frac{1}{2} r \cosh \theta \partial_\theta. \end{aligned} \quad (40)$$

We remind the reader that the variables  $r$  and  $\theta$  are defined in Eq.(15).

Writing  $L_{X^I} g_{ij} = 2C_I(r, \theta) g_{ij}$  we find the conformal factors of the CKVs  $X^I$   $I = 1, \dots, 6$  above in terms of the the conformal function. The result is:

$$C_1(r, \theta) = -\frac{1}{r} \sinh \theta \frac{N_{,\theta}}{N}, \quad C_2(r, \theta) = -\frac{1}{r} \cosh \theta \frac{N_{,\theta}}{N}$$

$$C_3(r, \theta) = \frac{N_{,\theta}}{N}, \quad C_4(r, \theta) = 1$$

$$C_5(r, \theta) = \frac{r}{2} \left( \frac{2N \cosh \theta + \sinh \theta N_\theta}{N} \right)$$

$$C_6(r, \theta) = \frac{r}{2} \left( \frac{2N \sinh \theta + \cosh \theta N_\theta}{N} \right).$$

We would like to remind the reader that the coupling function  $N(\theta)$  does not obey Eq.(17), otherwise the kinetic metric of the Lagrangian (18) is flat ( $R_{KM}$  vanishes) and we return to Sec. 3. The latter means that the vectors  $X^I$   $I = 1, \dots, 6$ , except the  $I = 4$ , are proper CKVs therefore they do not give (if proper) a Noether point symmetry. The vector  $X_4$  is a non-gradient HV which also does not produce a Noether point symmetry. Therefore, according to theorem in [50, 51], only Killing vectors are possible to serve as Noether symmetries. Killing vectors do not exist in general but only for special forms of the conformal function  $N(\theta)$ . Each of such forms of  $N(\theta)$  results in a potential  $V(\theta)$  hence in a scalar field potential which admits Noether point symmetries. In the following, we shall determine the possible  $N(\theta)$  forms which lead to a KV and give the corresponding Noether point symmetry and the corresponding Noether integral which will be used for the solution of the field equations.

##### 4.1. Searching for Noether symmetries

1. If  $N(\theta) = \frac{N_0}{\cosh 2\theta - 1}$  then  $X^5$  is a non-gradient KV and a Noether symmetry of the Lagrangian (18) for the potential

$$V(\theta) = \frac{V_0}{\cosh 2\theta - 1} \text{ or } V(\theta) = 0. \quad (41)$$

The corresponding Noether integral is

$$I_{X^5} = \frac{N_0^2 r^2}{(\cosh 2\theta - 1)^2} \left( r \dot{\theta} \sinh \theta - \dot{r} \cosh \theta \right). \quad (42)$$

2. If  $N(\theta) = \frac{N_0}{\cosh 2\theta + 1}$  then  $X^6$  is a non gradient KV,  $X^6$  and a Noether symmetry for the Lagrangian (18) if

$$V(\theta) = \frac{V_0}{\cosh 2\theta + 1} \text{ or } V(\theta) = 0. \quad (43)$$

The corresponding Noether integral is

$$I_{X^6} = \frac{N_0^2 r^2}{(\cosh 2\theta + 1)^2} \left( r\dot{\theta} \cosh \theta - \dot{r} \sinh \theta \right). \quad (44)$$

3. If  $N(\theta) = \frac{N_0}{\cosh^2(\theta + \theta_0)}$  then the linear combination  $X^{56} = c_1 X^5 + c_2 X^6$  where  $c_1 = \sinh(\theta_0)$  and  $c_2 = \cosh(\theta_0)$ .  $X^{56}$  is a Noether symmetry for the Lagrangian (18) if

$$V(\theta) = \frac{V_0}{\cosh^2(\theta + \theta_0)} \text{ or } V(\theta) = 0 \quad (45)$$

with Noether integral

$$I_{X^{56}} = \frac{N_0^2 r^2}{\cosh^4(\theta + \theta_0)} \left[ r\dot{\theta} \cosh(\theta + \theta_0) - \dot{r} \sinh(\theta + \theta_0) \right]$$

Obviously the third case is the most general situation and it contains cases 1 and 2 (and the trivial case) as special cases. Therefore, in the following, we look for analytic solutions for the vector  $X^{56}$  only.

We recall that  $\frac{1}{\sqrt{-2F(\theta)}} = N^2(\theta)$  from which follows:

$$F(\theta) = -\frac{1}{2N_0^4} \cosh^8(\theta + \theta_0), \quad N_0 \in \mathbb{R}. \quad (46)$$

We may consider  $\theta_0 = 0$  (e.g. by introducing the new variable  $\Theta = \theta + \theta_0$ ).

For the potential (45) Lagrangian (18) becomes

$$L = \frac{N_0^2}{\cosh^4 \theta} \left( -\frac{1}{2} \dot{r}^2 + \frac{1}{2} r^2 \dot{\theta}^2 \right) - r^2 \frac{V_0}{\cosh^2 \theta} \quad (47)$$

and the Hamiltonian

$$E = \frac{N_0^2}{\cosh^4 \theta} \left( -\frac{1}{2} \dot{r}^2 + \frac{1}{2} r^2 \dot{\theta}^2 \right) + r^2 \frac{V_0}{\cosh^2 \theta}. \quad (48)$$

The Euler-Lagrange equations provide the equations of motion:

$$\ddot{r} + r\dot{\theta}^2 - 4 \tanh \theta \dot{r}\dot{\theta} - 2 \frac{V_0}{N_0^2} r \cosh^2 \theta = 0 \quad (49)$$

$$\ddot{\theta} - 2 \tanh \theta \left( \frac{1}{r^2} \dot{r}^2 + \dot{\theta}^2 \right) + \frac{2}{r} \dot{r}\dot{\theta} - 2 \frac{V_0}{N_0^2} \cosh \theta \sinh \theta = 0 \quad (50)$$

and the Noether integral  $I$  for  $\theta_0 = 0$  becomes:

$$I = \frac{N_0^2 r^2}{\cosh^4(\theta + \theta_0)} \left[ r\dot{\theta} \cosh \theta - \dot{r} \sinh \theta \right]. \quad (51)$$

In order to proceed with the solution of the system of equations (49), (50) we change to the coordinates  $x, y$  which we define by the relations

$$r = \frac{x}{\sqrt{1 - x^2 y^2}}, \quad \theta = \arctan h(xy). \quad (52)$$

In the coordinates  $(x, y)$  the Lagrangian and the Hamiltonian are written as

$$L = \frac{N_0^2}{2} (-\dot{x}^2 + x^4 \dot{y}^2) - V_0 x^2 \quad (53)$$

$$E = \frac{N_0^2}{2} (-\dot{x}^2 + x^4 \dot{y}^2) + V_0 x^2 \quad (54)$$

and the Noether integral is

$$I = x^4 \dot{y}. \quad (55)$$

In the new variables, the Euler-Lagrange equations read:

$$\ddot{x} + 2x^3 \dot{y}^2 - \frac{2V_0}{N_0^2} x = 0 \quad (56)$$

$$\ddot{y} + \frac{4}{x} \dot{x} \dot{y} = 0. \quad (57)$$

In this context, from the Noether integral, we have

$$\dot{y} = \frac{I}{x^4} \quad (58)$$

which, upon substitution in the field equations, gives the system:

$$\ddot{x} + \frac{2I^2}{x^5} - \frac{2V_0}{N_0^2} x = 0 \quad (59)$$

$$\frac{N_0^2}{2} \left( -\dot{x}^2 + \frac{I^2}{x^4} \right) + V_0 x^2 = E. \quad (60)$$

from which we compute

$$\dot{x} = \sqrt{\frac{I^2}{x^4} + \frac{2V_0}{N_0^2} x^2 - \frac{2E}{N_0^2}} \quad (61)$$

and the analytical solution

$$\int \frac{dx}{\sqrt{\frac{I^2}{x^4} + \frac{2V_0}{N_0^2} x^2 - \frac{2E}{N_0^2}}} = t - t_0. \quad (62)$$

Also, integrating Eq.(58), we find

$$y(t) - y_0 = \int \frac{I}{x^4} dt. \quad (63)$$

If we consider the special case where  $I = 0$  then the analytic solution is

$$x = x_0 \sinh \left( \frac{\sqrt{2V_0}}{N_0} t + x_1 \right), \quad y = y_0 \quad (64)$$

with the Hamiltonian constrain  $E = -x_0^2 V_0$ .

4. Finally, if  $V_0 = 0$  (i.e. free particle) and  $I = 0$  the analytic solution becomes

$$x = x_0 t + x_1, \quad y = y_0 \quad (65)$$

with Hamiltonian constrain  $E = -\frac{1}{2} x_0^2 N_0^2$ .

## 5. CONCLUSIONS

In this work we have identified the Noether point symmetries of the equations of motion in the context of scalar-tensor cosmology considering a 2-dimensional minisuperspace  $\mathcal{Q} = \{\psi, a\}$ . We find that there is a rather large class of hyperbolic and exponential potentials which admit extra (beyond the  $\partial_t = 0$ ) Noether symmetries which lead to integral of motions. This approach is extremely efficient in physical problems since it can be utilized in order to simplify a given system of differential equations as well as to determine the integrability of the system. Based on the above arguments, we manage to provide general analytical solutions in scalar-tensor cosmologies assuming a FRW spatially flat metric. These solutions can be used in order to compare cosmographic parameters, such as the Hubble expansion rate, the deceleration parameter, snap, jerk and density parameters with observations [53]. Such an analysis is in progress and it will be published in a forthcoming paper.

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## Appendix A: Maximally symmetric space: the case $|k| = 1$

In order to complete Sec.3, we provide here the main steps of the Noether algebra in the case where  $k = \pm 1$ . Briefly, we start with the KVs of the kinematic metric

$$K^1 = \frac{1}{N_0^2} \frac{e^{-2k\theta}}{r} \left( k\partial_r + \frac{1}{r} \partial_\theta \right), \quad K^2 = \frac{1}{N_0^2} (-kr\partial_r + \partial_\theta),$$

$$K^3 = -r [\ln(re^{-k\theta}) - 1] \partial_r + \ln(re^{-k\theta}) \partial_\theta$$

where the vectors  $K^{1,2}$  are gradient and  $K^3$  is non-gradient. Also the HV is given by

$$H^i = \frac{1}{4} r [2 \ln(re^{-k\theta}) + 3] \partial_r - \frac{1}{2} \left[ \ln(re^{-k\theta}) + \frac{1}{2} \right] \partial_\theta.$$

Using the theorem in [50, 51] and making some simple calculations (see Sec. 3) we find the following results:

### 1. Noether symmetries generated by the KV $K^1$ .

a) If  $V(\theta) = V_0 e^{-2k\theta}$  then we have the Noether symmetries  $K^1$ ,  $tK^1$  with Noether integrals

$$I'_1 = \frac{d}{dt} (k\theta - \ln r), \quad I'_2 = t \left[ \frac{d}{dt} (k\theta - \ln r) \right] - (k\theta - \ln r)$$

b) If  $V(\theta) = V_0 e^{-2k\theta} - \frac{1}{4} p e^{2k\theta}$  then we have the Noether symmetries  $K^1$ ,  $tK^1$  with Noether integrals

$$I_1 = \frac{d}{dt} (k\theta - \ln r) - pt$$

$$I_2 = t \left[ \frac{d}{dt} (k\theta - \ln r) \right] - (k\theta - \ln r) - \frac{1}{2} p t^2$$

### 2. Noether symmetries generated by the KV $K^2$ .

a) If  $V(\theta) = V_0 e^{2k\theta}$  then we have the extra Noether symmetries  $K^2$ ,  $tK^2$  with Noether integrals

$$J_1 = \left[ \frac{d}{dt} \left( \frac{1}{2} e^{2k\theta} r^2 \right) \right], \quad J_2 = t \left[ \frac{d}{dt} \left( \frac{1}{2} e^{2k\theta} r^2 \right) \right] - \frac{1}{2} e^{2k\theta} r^2$$

b) If  $V(\theta) = (V_0 e^{2k\theta} - \frac{m}{2} k \theta e^{2k\theta})$ , then we have the Noether symmetries  $e^{\pm\sqrt{m}t} K^2$  with Noether integrals

$$J'_{1,2} = e^{\pm\sqrt{m}t} \left( \left[ \frac{d}{dt} \left( \frac{1}{2} e^{2\theta} r^2 \right) \right] \mp \frac{\sqrt{m}}{2} e^{2\theta} r^2 \right)$$

### 3. If $V(\theta) = 0$ then the system becomes the free particle (see [50]).

To this end the corresponding analytical solutions can be found utilizing the above integrals the arguments of Sec. 3 and the new coordinates  $(u, v) = (k\theta - \ln r, \frac{1}{2} e^{2k\theta} r^2)$ .

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